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Generation of waves by a moving plate in stratified shallow water

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Abstract

The generation of waves inside an ideal two-layer stratified shallow water by the uniform motion of a vertical plate partially immersed in the fluid mass is studied in two dimensions. The fluid is assumed to occupy an infinite channel of constant depth. Two distinctive cases are studied according to whether the submerged part of the moving plate is smaller or greater than the upper layer's depth. In the first case, the lower fluid layer is not influenced by the motion of the plate up to the second order of approximation and local perturbations, only, are created in the upper layer. For the second case, progressive waves of the first order are shown in both layers besides local perturbations of the second order in the lower layer only. Passing to the limit of homogeneous fluids, local perturbations only remain. This passage to the limit shows that the stratification of the fluid mass is significant for the generation progressive waves. The systems of stream lines are drawn for stratified and homogeneous fluids. © 1997 Elsevier Science B.V. All rights reserved.

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1. Introduction

Propagation of ocean and sea waves, due to the motion of ships or submarines, has been modelled and studied in several theoretical and experimental works. The model simulating this physical problem consists of a moving solid body of certain geometry, submerged or partially submerged in a fluid mass with a free surface.

For experimental manipulations, procedures and results, one can consult [2] and the references included.

The theoretical problem is nonlinear three-dimensional free boundary value problem constrained, in certain cases, by initial conditions (see, [11]).

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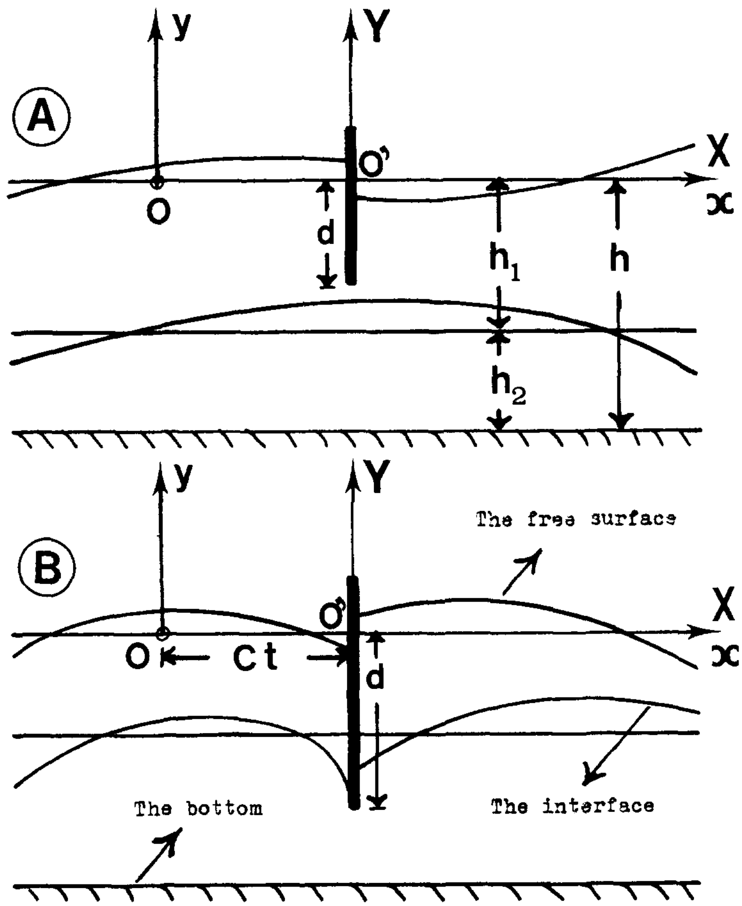

Fig. 1. The model and the frame of reference. (A) and (B) illustrate the two cases where the lower end of the plate lies within the upper and lower fluid layer, respectively.

The mathematical treatment is considerably simplified when the problem is studied in two dimensions for an incompressible and inviscid fluid.

The two-dimensional problem, within the frame of the linearized theorem of motion has been investigated by several authors for different configurations of the moving body (e.g., [9] for numerical study and [3] for theoretical study). However, this theorem is inadequate to describe the important non-linear aspects of the wave motion and it fails to describe long-wave propagation.

Different numerical techniques were developed to solve the non-linear system of equations and conditions to which the original problem is reduced (e.g. [10, 5]).

Imposing additional simplifying physical or geometrical conditions, certain analytical techniques were used by several authors to investigate the non-linear problem (e.g., [7, 6]).

For a better description of the physical problem, we investigate here the propagation of waves generated by the uniform horizontal motion of a vertical plate inside a double layer fluid mass. The stratified fluid is assumed to occupy an infinite channel of constant depth, and the velocity of the plate is supposed to be of the second order of a certain small parameter (ε). We considered two cases, according to whether the plate is smaller or bigger than the upper layers's depth (see Fig. 1).

Euler's description is used and the problem is studied within the frame of the two-dimensional shallow-water theory. The free surface and the interface are assumed to be always near their positions at rest. Convenient double-series representations of asymptotic nature for the unknown functions of the problem are used. These representations are slightly different from those used by Abou-Dina and Helal [1] for a flow past a fixed barrier.

In the first case (Fig. 1(a)) where the moving plate does not reach the lower layer, this layer is not affected by the motion of the plate and it remains at rest up to the second order of approximation. A local disturbance of the second order appears in the upper layer in the region neighbouring the moving plate.

In the second case (Fig. 1(b)), the submerged part of the plate is greater than the upper layer's depth. The fluid particles of this layer are found to have the same displacement as that of the plate. In the lower fluid layer barotropic and baroclinic progressive waves of the first order and local perturbations of the second order are created.

In the limiting case of a homogeneous fluid, the flow is quite similar to that in the upper layer of the first case mentioned above. This indicates the important influence of the stratification of the fluid mass and the relative dimensions of the moving plate on the generation of progressive waves.

As a demonstration, the systems of stream lines are drawn, in a frame of reference moving with the plate, for both cases of stratified fluid and also in the case of a homogeneous fluid.

2. Problem and frame of reference

A fluid mass consisting of two layers of immiscible and ideal liquids of constant densities occupies an infinite channel of finite and constant depth. A vertical thin plate, partially immersed in the fluid mass, moves horizontally with a constant velocity (C). The problem is to determine the upstream and the downstream waves propagating in the fluid mass.

The Cartesian frame of reference $O(x,y)$ shown in Fig. 1, with origin at the initial position of the plate, is used. Another frame of reference $O'(X,Y)$, fixed in the plate with origin at O' and: $X = x - Ct$, $Y = y$, will be also used in the demonstrations.

3. Notation

The following notation is used throughout this paper.

$C = \varepsilon^2 C_0$	velocity of the plate
d	length of the submerged part of the plate (Fig. 1)
g	acceleration of gravity
h_1, h_2	constant depths of the upper and lower
$(h = h_1 + h_2)$	layers of the fluid, respectively (Fig. 1)
$P(x,y,t)$	pressure
$P_m(z)$	Legendre polynomial of order m with argument z
t	time
$\mathbf{w}(x,y,t)$	velocity of the particle occupying the position (x,y) at the instant t
$\Psi(x,y,t)$	stream function

$\Phi(x,y,t)$	velocity potential
$y=\eta^{(1)}(x,t)$	equations of the free surface and the
$y=\eta^{(2)}(x,t)$	interface, respectively
δ_{nm}	kroneker delta $=1$ if $n=m$; 0 if $n\neq m$
ε	a small parameter
ρ_1, ρ_2	constant densities of the upper and lower layers, respectively ($\rho_2>\rho_1$)
$\partial_{xy\ldots z}$	an operator of partial differentiation w.r.t. variables $x, y, \ldots$ and z.

Superscripts

1,2	upper and lower layers, respectively
$+,-$	regions on the right and on the left of the plate, respectively

4. Equations of motion and the shallow water theory

As in [1], we use the set of distorted variables $\hat{x}$, $\hat{y}$, $\hat{t}$ defined in terms of x, y, t by

$$\hat{x}=\varepsilon x, \quad \hat{y}=y, \quad \hat{t}=\varepsilon t, \tag{1}$$

where ε is a small parameter.

We write $\hat{\Phi}^{(i)}$, $\hat{P}^{(i)}$, $\hat{\Psi}^{(i)}$ and $\hat{\eta}^{(i)}$ for the functions $\Phi(i)$ $(\hat{x}/\varepsilon, \hat{y}, \hat{t}/\varepsilon)$, $P^{(i)}(\hat{x}/\varepsilon, \hat{y}, \hat{t}/\varepsilon)$, $\Psi^{(i)}(\hat{x}/\varepsilon, \hat{y}, \hat{t}/\varepsilon)$ and $\eta^{(i)}(\hat{x}/\varepsilon, \hat{t}/\varepsilon)$, respectively, where i hereafter stands for both 1 and 2.

The system of equations and conditions of the problem is written in terms of the distorted variables $\hat{x}$, $\hat{y}$, $\hat{t}$ as

(i) In the fluid layers

$$\varepsilon^2\partial_{\hat{x}\hat{x}}\hat{\Phi}^{(i)} + \partial_{\hat{y}\hat{y}}\hat{\Phi}^{(i)} = 0, \tag{2}$$

$$\hat{P}^{(i)}(\hat{x},\hat{y},\hat{t}) = \rho_i[\hat{H}(\hat{t})\delta_{i2} + \varepsilon\partial_{\hat{t}}\hat{\Phi}^{(i)} + (\tfrac{1}{2})\{\varepsilon^2(\partial_{\hat{x}}\hat{\Phi}^{(i)})^2 + \partial_{\hat{y}}\hat{\Phi}^{(i)})^2\} + g\hat{y}], \tag{3}$$

where $\hat{H}(\hat{t})$ is an arbitrary function of $\hat{t}$ to be determined.

(ii) On the free surface, the impermeability implies

$$\partial_{\hat{y}}\hat{\Phi}^{(1)} = \varepsilon^2\partial_{\hat{x}}\hat{\eta}^{(1)}\partial_{\hat{x}}\hat{\Phi}^{(1)} + \varepsilon\partial_{\hat{t}}\hat{\eta}^{(1)}, \quad \text{at } \hat{y}=\hat{\eta}^{(1)}(\hat{x},\hat{t}), \tag{4}$$

and the isobarity gives

$$\varepsilon\partial_{\hat{t}}\hat{\Phi}^{(1)} + (\tfrac{1}{2})[\varepsilon^2(\partial_{\hat{x}}\hat{\Phi}^{(1)})^2 + (\partial_{\hat{y}}\hat{\Phi}^{(1)})^2]g\hat{y} = 0, \quad \text{at } \hat{y}=\hat{\eta}^{(1)}(\hat{x},\hat{t}). \tag{5}$$

(iii) At the interface, the impermeability leads to

$$\partial_{\hat{y}}\hat{\Phi}^{(i)} = \varepsilon^2\partial_{\hat{x}}\hat{\eta}^{(2)} + \partial_{\hat{x}}\hat{\Phi}^{(i)} + \varepsilon\partial_{\hat{t}}\hat{\eta}^{(2)}, \quad \text{at } \hat{y}=\hat{\eta}^{(2)}(\hat{x},\hat{t}) \tag{6}$$

and the continuity of the pressure through this boundary yields

$$\begin{aligned}&\rho_1\{g\hat{\eta}^{(2)} + \varepsilon\partial_{\hat{t}}\Phi^{(1)} + (\tfrac{1}{2})[\varepsilon^2(\partial_{\hat{x}}\hat{\Phi}^{(1)})^2 + (\partial_{\hat{y}}\hat{\Phi}^{(1)})^2]\}\\ &\quad = \rho_2\{g\hat{\eta}^{(2)} + \varepsilon\partial_{\hat{t}}\hat{\Phi}^{(2)}\\ &\qquad + (\tfrac{1}{2})[\varepsilon^2(\partial_{\hat{x}}\hat{\Phi}^{(2)})^2 + (\partial_{\hat{y}}\hat{\Phi}^{(2)})^2] + \hat{H}(\hat{t})\} \quad \text{at } \hat{y}=\hat{\eta}^{(2)}(\hat{x},\hat{t}).\end{aligned} \tag{7}$$

(iv) At the bottom of the channel, the impermeability gives

$$\partial_{\hat{y}}\hat{\Phi}^{(2)} = 0 \quad \text{at } \hat{y} = -h. \tag{8}$$

(v) On the plate, the condition is

$$\partial_{\hat{x}}\hat{\Phi}^{(j)} = \varepsilon C_0 \quad \text{at} \quad \hat{x} = \varepsilon^2 C_0 \hat{t}, \ -d \leqslant \hat{y} \leqslant 0. \tag{9}$$

Also, the dynamical quantities are continuous for $\hat{x} = \varepsilon^2 C_0 \hat{t}$, $-h \leqslant \hat{y} \leqslant -d$, where j stands for 1 in case (A) and for 1 and 2 in case (B) (see Fig. 1).

(vi) The radiation condition states that no waves come from infinity.

(vii) The Cauchy–Riemann conditions give

$$\varepsilon \partial_{\hat{x}}\hat{\Phi}^{(i)} = \partial_{\hat{y}}\hat{\Psi}^{(i)}, \tag{10}$$

$$\partial_{\hat{y}}\hat{\Phi}^{(i)} = -\varepsilon \partial_{\hat{x}}\hat{\Psi}^{(i)}. \tag{11}$$

In the light of the results of Abou-Dina and Helal [1], for stratified fluids, we use the following double-series representations for the unknown functions of the present problem:

$$\hat{\Phi}^{(i)}(\hat{x}, \hat{y}, \hat{t}) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \varepsilon^n \exp(-m\pi\,|\hat{x} - \varepsilon^2 C_0 \hat{t}|/\varepsilon h_i)\hat{\Phi}^{(i)}_{n,m}(\hat{x}, \hat{y}, \hat{t}), \tag{12a}$$

$$\hat{\eta}^{(i)}(\hat{x}, \hat{t}) = -h_1\delta_{i2} + \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \varepsilon^n \exp(-m\pi\,|\hat{x} - \varepsilon^2 C_0 \hat{t}|/\varepsilon h_i)\hat{\eta}^{(i)}_{n,m}(\hat{x}, \hat{t}), \tag{12b}$$

$$\hat{\Psi}^{(i)}(\hat{x}, \hat{y}, \hat{t}) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} \varepsilon^n \exp(-m\pi\,|\hat{x} - \varepsilon^2 C_0 \hat{t}|/\varepsilon h_i)\hat{\Psi}^{(i)}_{n,m}(\hat{x}, \hat{y}, \hat{t}). \tag{12c}$$

A similar expansion is used by Germain [4], in the case of homogeneous fluids, and is shown to be asymptotically convergent.

For simplification, in the following, the hats (ˆ) over the symbols will be omitted.

According to the shallow-water theory formulated by Germain [4], the system of equations and conditions (2) to (11) must be verified at each order (n, m) of the expressions (12). Application of this procedure leads to the following expression for the velocity potential up to the second order of the small parameter (ε):

$$\begin{aligned}\Phi^{(i)\pm}(x, y, t) = {}& \varepsilon[A^{(i)\pm}(x \mp C_1 t) + B^{(i)\pm}(x \mp C_2 t)] + \varepsilon^2[R^{(i)\pm}(x \mp C_1 t) + S^{(i)\pm}(x \mp C_2 t) \\ & + \sum_{m=1}^{\infty} A^{(i)\pm}_{2,m}(t)\exp(-m\pi\,|x - \varepsilon^2 C_0 t\,|/\varepsilon h_i)\cos(m\pi\{y + \delta_{i2}h\}/h_i)] \\ & + O(\varepsilon^3),\end{aligned} \tag{13}$$

where

$$C_i^2 = (g/2)[h - (-1)^i((h_1 - h_2)^2 + 4(\rho_1/\rho_2)h_1 h_2)^{1/2}]. \tag{14a}$$

Here $C_1 > C_2$ the wave speed C_1 corresponds to the external (or barotropic) modes, and C_2 corresponds to the internal (or baroclinic) modes. The functions $A^{(i)\pm}$, $B^{(i)\pm}$ and $A^{(i)\pm}_{2,m}$ are arbitrary functions of their arguments to be determined, with

$$A^{(1)\pm}(x \mp C_1 t) = \gamma_1 A^{(2)\pm}(x \mp C_1 t), \tag{14b}$$

$$B^{(1)\pm}(x \mp C_2 t) = \gamma_2 B^{(2)\pm}(x \mp C_2 t) \tag{14c}$$

and

$$\gamma_i = gh_2/(C_i^2 - gh_1). \tag{14d}$$

The functions $R^{(i)\pm}$, $S^{(i)\pm}$ of expression (13) are arbitrary functions combined by relations similar to those in Eqs. (14) in terms of $A^{(i)\pm}$, $B^{(i)\pm}$, respectively.

The corresponding expressions for the free surface and the interface are given by

$$\eta^{(1)\pm}(x,t) = -(\varepsilon/g)\partial_t \Phi^{(1)\pm}(x,y,t) \quad \text{at } y = 0, \tag{15a}$$

$$\eta^{(2)\pm}(x,t) = -h_1 + (\varepsilon/g)(\rho_2 - \rho_1)\partial_t[\rho_1 \Phi^{(1)\pm} - \rho_2 \Phi^{(2)\pm}] \quad \text{at } y = -h_1, \tag{15b}$$

respectively.

The velocity potential given by (13) satisfies all the conditions of the problem except that on the plate given by (9). Two distinctive cases accordingly arise:

(A) The case where the submerged part of the plate is completely contained in the upper fluid layer (Fig. 1(a)).

(B) The case where the plate is sufficiently long to reach the lower fluid layer (Fig. 1(b)).

In both cases, the pressure is continuous on the bottom of the channel. This condition, applied at $x = \varepsilon^2 C_0 t$, $y = -h$, together with expressions (3) and (13) give at the first order of ε

$$A^{(2)+}(-C_1 t) + B^{(2)+}(-C_2 t) - A^{(2)-}(C_1 t) - B^{(2)-}(C_2 t) = 0. \tag{16}$$

For the derivation of expression (16) from the continuity of the pressure, use is made of the Taylor expansion of functions in powers of the small parameter ε, and the first terms only are kept. The same procedure will be used, in the following analysis, when a condition is applied at the plate level $(x = \varepsilon^2 C_0 t)$.

5. Cases

5.1. Case (A)

In the case where the plate is too short to close the upper fluid layer, the interface is continuous. Applying this condition at $x = \varepsilon^2 C_0 t$, and using expressions (13)–(15), we get

$$\begin{aligned}(\rho_2 - \rho_1\gamma_1)A^{(2)+}(-C_1 t) + (\rho_2 - \rho_1\gamma_2)B^{(2)+}(-C_2 t)\\ = -(\rho_2 - \rho_1\gamma_1)A^{(2)-}(C_1 t) - (\rho_2 - \rho_1\gamma_2)B^{(2)-}(C_2 t),\end{aligned} \tag{17}$$

also, in this case the velocity ($w^{(i)} = \nabla\Phi^{(i)}$) is continuous in the fluid mass. This is written at $x = \varepsilon^2 C_0 t$ as

$$\partial_x \phi^{(1)-} = \partial_x \Phi^{(1)+}, \quad -h_1 \leqslant y \leqslant -d, \tag{18a}$$

$$\partial_y \phi^{(1)-} = \partial_y \Phi^{(1)+}, \quad -h_1 \leqslant y \leqslant -d, \tag{18b}$$

$$\partial_x \phi^{(2)-} = \partial_x \Phi^{(2)+}, \quad -h \leqslant y \leqslant -h_1, \tag{18c}$$

$$\partial_y \phi^{(2)-} = \partial_y \Phi^{(2)+}, \quad -h \leqslant y \leqslant -h_1. \tag{18d}$$

The boundary condition on the plate takes the form

$$\partial_x \Phi^{(1)-} = \partial_x \Phi^{(1)+}, = \varepsilon C_0 t \quad \text{at } x = \varepsilon^2 C_0 t, -d \leqslant y \leqslant 0. \tag{19}$$

Application of (18a) and (19) to (13) leads with (14) to

$$(\gamma_1/C_1)A^{(2)+}(-C_1 t) + (\gamma_2/C_2)B^{(2)+}(-C_2 t)$$
$$+ (\gamma_1/C_1)A^{(2)-}(C_1 t) + (\gamma_2/C_2)B^{(2)-}(C_2 t) = 0. \tag{20}$$

Relation (18c) and expression (13) give at $x = \varepsilon^2 C_0 t$

$$(1/C_1)A^{(2)+}(-C_1 t) + (1/C_2)B^{(2)+}(-C_2 t)$$
$$+ (1/C_1)A^{(2)-}(C_1 t) + (1/C_2)B^{(2)-}(C_2 t) = 0. \tag{21}$$

The system of Equations (16), (17), (20) and (21) has the solution

$$A^{(2)\pm}(\mp C_1 t) = B^{(2)\pm}(\mp C_2 t) = 0,$$

which together with relations (14) give

$$A^{(1)\pm}(x \mp C_1 t) = A^{(2)\pm}(x \mp C_1 t)$$
$$= B^{(1)\pm}(x \mp C_2 t) = B^{(2)\pm}(x \mp C_2 t) = 0. \tag{22}$$

Relations (18c) and (18d) applied to expression (13) lead to

$$A_{2,m}^{(2)\pm}(t) = 0, \quad m = 1, 2, 3, \ldots\,. \tag{23}$$

The coefficients $A_{2,m}^{(1)\pm}(t)$ are found by (13), (18a) and (19) to be related for $m = 1, 2, 3, \ldots$ by

$$-A_{2,m}^{(1)+}(t) = A_{2,m}^{(1)-}(t) = (h_1/\pi)a_m(t), \tag{24}$$

where the functions $a_m(t)$, $m = 1, 2, 3, \ldots$, can be seen by (13), (18a), (18b), (19) and (24) to satisfy the following dual series equations

$$-C_0 + \sum_{m=1}^{\infty} m a_m(t) \cos(m\pi y/h_1) = 0, \quad -d \leqslant y \leqslant 0, \tag{25a}$$

$$\sum_{m=1}^{\infty} m a_m(t) \sin(m\pi y/h_1) = 0, \quad -h_1 \leqslant y \leqslant -d. \tag{25b}$$

The linearity of Eqs. (25) shows that $a_m(t)$, $m=1,2,3,\dots$, are constants. The solution of this system of equations is given using [8] in the form

$$a_m = -C_0\left(\frac{P_m(\delta_1)-P_{m-1}(\delta_1)}{m}\right) \tag{26a}$$

with

$$\delta_1 = \cos(\pi d/h_1). \tag{26b}$$

Using (13), (22)–(24) and (26), we get the velocity potential in the fluid mass in the form

$$\Phi^{(1)\pm}(x,y,t) = \pm(\varepsilon^2 h_1 C_0/\pi)\sum_{m=1}^{\infty}\left(\left(\frac{P_m(\delta_1)-P_{m-1}(\delta_1)}{m}\right)\right.$$
$$\left.\times\exp(-m\pi\,|x-\varepsilon^2 C_0 t|/\varepsilon h_1)\cos(m\pi y/h_1)\right) + \mathrm{O}(\varepsilon^3) \tag{27a}$$

$$\Phi^{(2)\pm}(x,y,t) = \mathrm{O}(\varepsilon^3). \tag{27b}$$

Hence, in the case where the plate is smaller than the upper fluid layer's depth, the lower layer is not affected by the motion of the plate and it remains at rest up to the second order of ε. The progressive waves disappear in the upper layer, and local oscillations only are found. These local perturbations vanish far from the plate and they have no contributions, up to the second order, to the elevations of the free surface and the interface.

The corresponding expressions of the stream functions are obtained using relations (10), (11) and (27) as

$$\Psi^{(1)\pm}(x,y,t) = -(\varepsilon^2 h_1 C_0/\pi)\sum_{m=1}^{\infty}\left(\left(\frac{P_m(\delta_1)-P_{m-1}(\delta_1)}{m}\right)\right.$$
$$\left.\times\exp(-m\pi\,|x-\varepsilon^2 C_0 t|/\varepsilon h_1)\sin(m\pi y/h_1)\right) + \mathrm{O}(\varepsilon^3), \tag{28a}$$

$$\Psi^{(2)\pm}(x,y,t) = \mathrm{O}(\varepsilon^3). \tag{28b}$$

In the frame of reference $\mathrm{O}'(X,Y)$, moving with the plate, the stream function $\bar{\Psi}(X,Y,t)$ is given using (28) as

$$\bar{\Psi}^{(1)\pm}(X,Y,t) = -C\left[Y + h_1\sum_{m=1}^{\infty}\left(\left(\frac{P_m(\delta_1)-P_{m-1}(\delta_1)}{m\pi}\right)\right.\right.$$
$$\left.\left.\times\exp(-m\pi\,|X|/h_1)\sin(m\pi Y/h_1)\right)\right] + \mathrm{O}(\varepsilon^3), \tag{29a}$$

$$\bar{\Psi}^{(2)\pm}(X,Y,t) = -CY + \mathrm{O}(\varepsilon^3). \tag{29b}$$

Expressions (29) show that the stream function, in the moving frame of reference, is steady and symmetric with respect to the variable X. It assumes the values $0, -Ch_1$ and $-Ch$ at the free

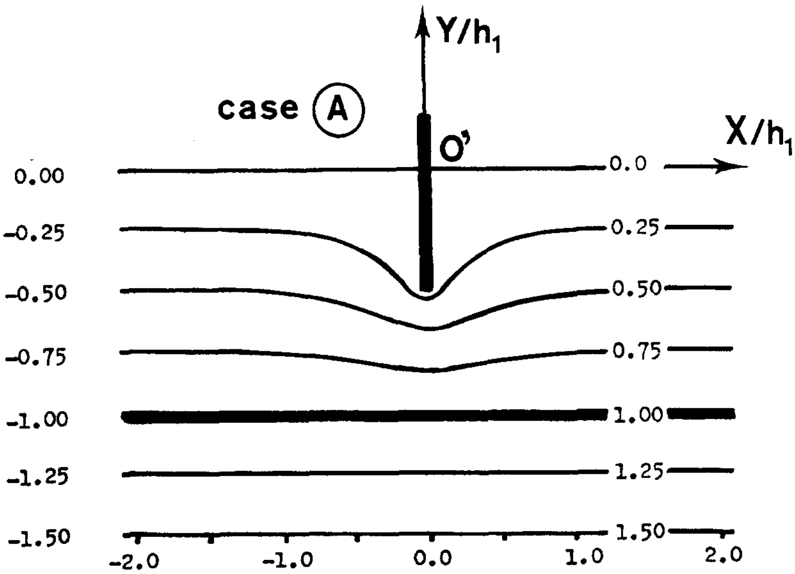

Fig. 2. The system of stream lines of case (A), drawn in the frame of reference fixed in the plate. The particular choice $d=h_2=h_1/2$ is used. The numbers assigned to each line refers to the corresponding value of the ratio $\bar{\Psi}(x,Y,t)/(h_1 C)$.

surface, the interface and the bottom of the channel, respectively. Any other value between these values corresponds to another stream line. In the upper fluid layer, the stream lines are straight lines.

As a demonstration, we give in Fig. 2 the system of stream lines drawn in the moving frame of reference for the particular choice of $d=h_2=h_1/2$. The effect of the moving plate is localized in the upper fluid layer in a region neighbouring the plate. In this region, the fluid particles are accelerated.

5.2. Case (B)

In this case (see Fig. 1(b)), the plate is long enough to close the upper fluid layer and to be partially immersed in the lower layer.

The conditions at the plate level ($x=\varepsilon^2 C_0 t$) are given as

$$\partial_x \Phi^{(1)-} = \partial_x \Phi^{(1)+} = \varepsilon C_0, \quad -h_1 \leqslant y \leqslant 0, \tag{30a}$$

$$\partial_x \Phi^{(2)-} = \partial_x \Phi^{(2)+} = \varepsilon C_0, \quad -d \leqslant y \leqslant -h_1, \tag{30b}$$

$$\partial_x \Phi^{(2)-} = \partial_x \Phi^{(2)+}, \quad -h \leqslant y \leqslant -d, \tag{30c}$$

$$\partial_y \Phi^{(2)-} = \partial_y \Phi^{(2)+}, \quad -h \leqslant y \leqslant -d. \tag{30d}$$

Conditions (30) together with expressions (13) and (14) give

$$-(\gamma_1/C_1)A^{(2)+}(-C_1 t) - (\gamma_2/C_2)B^{(2)+}(-C_2 t) = C_0 t, \tag{31a}$$

$$(\gamma_1/C_1)A^{(2)-}(C_1 t) + (\gamma_2/C_2)B^{(2)-}(C_2 t) = C_0 t, \tag{31b}$$

$$\begin{aligned}(1/C_1)&A^{(2)+}(-C_1 t) + (1/C_2)B^{(2)+}(-C_2 t)\\ &+ (1/C_1)A^{(2)-}(C_1 t) + (1/C_2)B^{(2)-}(-C_2 t) = 0.\end{aligned} \tag{31c}$$

Using relations (14), the solution of the system of equations (16) and (31) leads to the following expressions:

$$A^{(2)+}(-C_1t) = -A^{(2)-}(c_1t) = (1/\gamma_1)A^{(1)+}(-C_1t)$$
$$= -(1/\gamma_1)A^{(1)-}(C_1t) = C_0C_1C_2t/(C_1\gamma_2 - C_2\gamma_1), \tag{32a}$$

$$B^{(2)+}(-C_2t) = -B^{(2)-}(c_2t) = (1/\gamma_2)B^{(1)+}(-C_2t)$$
$$= -(1/\gamma_2)B^{(1)-}(C_2t) = -C_0C_1C_2t/(C_1\gamma_2 - C_2\gamma_1). \tag{32b}$$

Also, we get from expressions (13) and (30) for $m = 1, 2, 3, \dots$.

$$A^{(1)+}_{2,m}(t) = A^{(1)-}_{2,m}(t) = 0, \tag{33a}$$

$$-A^{(2)+}_{2,m}(t) = A^{(2)-}_{2,m}(t) = (h_2/\pi)b_m(t), \tag{33b}$$

where the coefficients $b_m(t)$ can be seen by (13), (30c, d), (31) and (33b) to satisfy the following dual series equations:

$$\left[\frac{(C_1 - C_2)}{(C_1\gamma_2 - C_2\gamma_1)} - 1\right] C_0 + \sum_{m=1}^{\infty} m b_m(t) \cos\left(\frac{m\pi}{h_2}(y+h)\right) = 0, \quad -d \leqslant y \leqslant -h_1,\ h_2 \neq 0, \tag{34a}$$

$$\sum_{m=1}^{\infty} m b_m(t) \sin\left(\frac{m\pi}{h_2}(y+h)\right) = 0, \quad -h \leqslant y \leqslant -d,\ h_2 \neq 0. \tag{34b}$$

The solution of system (34) is obtained using [8] in the form

$$b_m = (-1)^m C_0 \left[\frac{(C_1 - C_2)}{(C_1\gamma_2 - C_2\gamma_1)} - 1\right]\left[\frac{P_m(\delta_2) - P_{m-1}(\delta_2)}{m}\right], \quad m = 1, 2, 3, \dots \tag{35a}$$

with

$$\delta_2 = \cos[\pi(d - h_1)/h_2], \quad h_2 \neq 0. \tag{35b}$$

Using expressions (13), (32), (33) and (35), the velocity potential is given, in this case, up to the second order of ε as

$$\Phi^{(1)\pm}(x, y, t) = \left(\frac{\varepsilon C_0}{(C_1\gamma_2 - C_2\gamma_1)}\right)\{-C_2\gamma_1[x \mp C_1t] + C_1\gamma_2[x \mp C_2t]\} + O(\varepsilon^3), \tag{36a}$$

$$\Phi^{(2)\pm}(x, y, t) = \left(\frac{\varepsilon C_0}{(C_1\gamma_2 - C_2\gamma_1)}\right)\{-C_2[x \mp C_1t]$$
$$+ C_1[x \mp C_2t]\} \mp \varepsilon^2\left(\frac{h_2C_0}{\pi}\right)\left[\frac{(C_1 - C_2)}{(C_1\gamma_2 - C_2\gamma_1)} - 1\right]$$
$$\times \sum_{m=1}^{\infty}\Bigg((-1)^m\left(\frac{P_m(\delta_2) - P_{m-1}(\delta_2)}{m}\right)\exp(-m\pi\,|x - \varepsilon^2C_0t|/\varepsilon h_2)$$
$$\times \cos[m\pi(y+h)/h_2]\Bigg) + O(\varepsilon^3). \tag{36b}$$

Therefore, in the case where the plate reaches the lower fluid layer, both layers are affected by its motion. Progressive waves of the first order are created in the two layers, and local oscillations of the second order, vanishing far from the plate, are present in the lower layer only. In this case, the contributions of the motion of the plate to the elevations of the free surface and the interface, up to the second order of ε, are constant terms given by expressions (15) and (36) in the form

$$\eta^{(1)\pm}(x,t) = \pm\left(\frac{\varepsilon^2}{g}\right)\left(\frac{C_0C_1C_2(\gamma_2-\gamma_1)}{(C_1\gamma_2-C_2\gamma_1)}\right)+\mathrm{O}(\varepsilon^3), \tag{37a}$$

$$\eta^{(2)\pm}(x,t) = -h_1 \mp \left(\frac{\varepsilon^2\rho_1}{g(\rho_2-\rho_1)}\right)\left(\frac{C_0C_1C_2(\gamma_2-\gamma_1)}{(C_1\gamma_2-C_2\gamma_1)}\right)+\mathrm{O}(\varepsilon^3). \tag{37b}$$

The stream function describing the resulting flow is given, up to the second order, by relations (10), (11) and (36) as

$$\Psi^{(1)\pm}(x,y,t) = \varepsilon^2C_0y+\mathrm{O}(\varepsilon^3), \tag{38a}$$

$$\begin{aligned}\Psi^{(2)\pm}(x,y,t) = \varepsilon^2C_0&\Bigg(\frac{(C_1-C_2)}{(C_1\gamma_2-C_2\gamma_1)}y\\ &+\left(\frac{h_2}{\pi}\right)\left\{\frac{(C_1-C_2)}{(C_1\gamma_2-C_2\gamma_1)}-1\right\}\sum_{m=1}^{\infty}\Bigg((-1)^m\left(\frac{P_m(\delta_2)-P_{m-1}(\delta_2)}{m}\right)\\ &\times\exp(-m\pi\,|x-\varepsilon^2C_0t|/\varepsilon h_2)\sin[m\pi(y+h)/h_2]\Bigg)\Bigg)+\mathrm{O}(\varepsilon^3).\end{aligned} \tag{38b}$$

In this case also, the stream function $\bar{\Psi}$ in the moving frame of reference is steady and symmetric with respect to the variable X, up to the second order, and is given by

$$\bar{\Psi}^{(1)\pm}(X,Y,t) = 0+\mathrm{O}(\varepsilon^2), \tag{39a}$$

$$\begin{aligned}\bar{\Psi}^{(2)\pm}(X,Y,t) = C&\left(\frac{C_1(1-\gamma_2)-C_2(1-\gamma_1)}{(C_1\gamma_2-C_2\gamma_1)}\right)\\ &\times\Bigg((Y+h_1)+\sum_{m=1}^{\infty}\Bigg((-1)^m\left(\frac{P_m(\delta_2)-P_{m-1}(\delta_2)}{m\pi}\right)\exp(-m\pi\,|X|/h_2)\\ &\times\sin[m\pi(Y+h)/h_2]\Bigg)\Bigg)+\mathrm{O}(\varepsilon^3).\end{aligned} \tag{39b}$$

In the moving frame of reference, the upper fluid layer is shown by expression (39a) to be fixed. The stream function $\bar{\Psi}$ varies continuously from zero to the value α between the interface and the

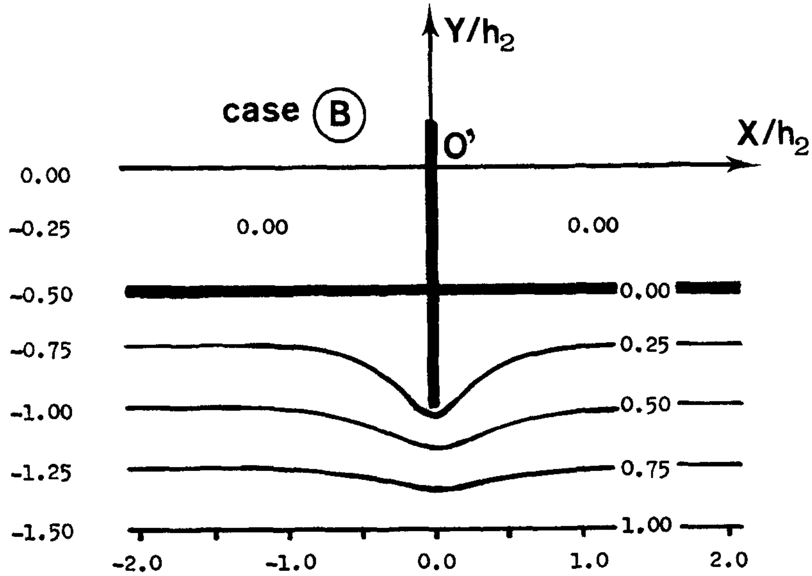

Fig. 3. The system of stream lines of case (B), drawn in the moving frame of reference. The particular choice $d = h_2 = 2h_1$ is considered. The numbers assigned to each line indicates the corresponding value of the ratio $\bar{\Psi}(X, Y, t)/(-\alpha)$, where α is given by expression (40).

bottom, respectively, with

$$\alpha = h_2 C[C_2(1-\gamma_1) - C_1(1-\gamma_2)]/(C_1\gamma_2 - C_2\gamma_1). \tag{40}$$

Fig. 3 exhibits the system of stream lines in the moving frame of reference for the particular choice of $d = h_2 = 2h_1$. The fluid is fixed in the upper layer. The stream lines in the lower layer are straight lines curved in the region neighbouring the plate where the fluid particles are accelerated.

6. Homogeneous fluids

In the limiting case when h_2 tends to zero and h_1 equals h (for case A) or h_1 tends to zero and h_2 equals h (for case B), the fluid occupying the channel is homogeneous with depth h. Both the relations (27a) and (36b) lead, separately, at the limit to the following expression for the velocity potential:

$$\Phi^{\pm}(x, y, t) = \pm(\varepsilon^2 h C_0/\pi) \sum_{m=1}^{\infty} \left(\left(\frac{P_m(\delta) - P_{m-1}(\delta)}{m} \right) \times \exp(-m\pi\,|x - \varepsilon^2 C_0 t|/\varepsilon h) \cos(m\pi y/h) \right) + \mathrm{O}(\varepsilon^3) \tag{41a}$$

with

$$\delta = \pi d/h. \tag{41b}$$

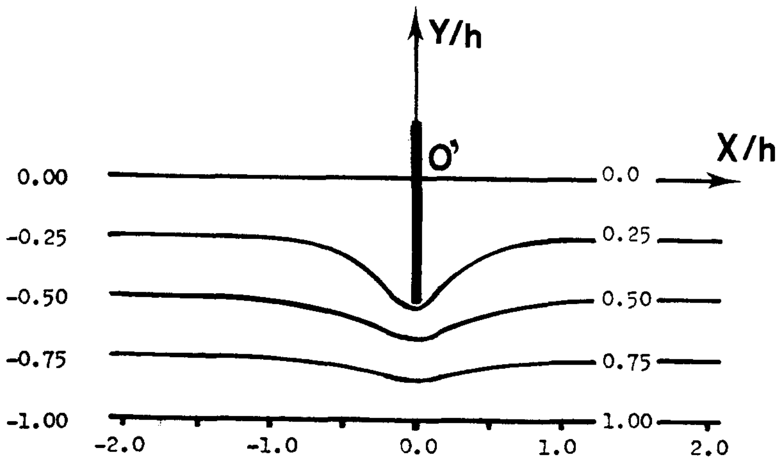

Fig. 4. The system of stream lines for the case of homogeneous fluid, drawn in the frame of reference moving with the plate. The length of the plate is one-half of the channel's depth. The number assigned to each line refers to the corresponding value of the ratio $\bar{\Psi}(X,Y,t)/(hC)$.

The corresponding stream function is given, in this case, using either (29a) or (39b) in the form

$$\Psi^{\pm}(x,y,t) = -(\varepsilon^2 hC_0/\pi)\sum_{m=1}^{\infty}\left(\left(\frac{P_m(\delta)-P_{m-1}(\delta)}{m}\right)\right.$$
$$\left.\times\exp(-m\pi\,|x-\varepsilon^2 C_0 t|/\varepsilon h)\sin(m\pi y/h)\right) + \mathrm{O}(\varepsilon^3). \tag{42}$$

Hence, for homogeneous fluids, local perturbations vanishing far from the moving plate are, only, found in the fluid mass. The free surface elevation is of the order higher than the second.

The results of the present and the preceding sections prove that the stratification of the fluid mass and the relative length of the plate play an important role in the generation of progressive waves by the motion of the plate.

Here also, the stream function $\bar{\Psi}$, in the moving frame of reference, is steady and symmetric with respect to the variable X and is given by

$$\bar{\Psi}^{\pm}(X,Y,t) = -C\left(Y + h\sum_{m=1}^{\infty}\left(\left(\frac{P_m(\delta)-P_{m-1}(\delta)}{m\pi}\right)\right.\right.$$
$$\left.\left.\times\exp(-m\pi\,|X|/h)\sin(m\pi y/h)\right)\right) + \mathrm{O}(\varepsilon^3). \tag{43}$$

Fig. 4 shows the resulting system of stream lines in the homogeneous layer, drawn in the moving frame of reference, for the particular choice of $d = h/2$. The disturbance is, readily, located near the plate and the fluid particles are accelerated there.

7. Comments and conclusive remarks

The model studied, here, reveals some of the features of the physical phenomena of ocean and sea waves generated by the uniform motion of ships and submarines where the horizontal extent of the moving bodies, in the direction of the motion, is small.

The small parameter ε is a criterion of the smallness of the horizontal extent of the moving bodies in the realistic problem (see [1]), and the velocity of these bodies is assumed to be of the second order of ε.

The mathematical procedure used enables us to separate progressive waves from local perturbations. Progressive waves of the first order and quadratic local perturbations are calculated in both cases of short and long moving plates, as given by expressions (27) and (36), respectively.

The nonsymmetry, on the two sides of the plate, of expressions (37) for the free surface and the interface elevations reconfirms the well-known result that when the Froude number is subcritical, the flow is non-symmetrical in the frame of the plate. The new result, here, is that the flow is symmetrical in the lower orders, and the non-symmetry appears in the higher orders. This is a result of the physical and geometrical limitations imposed to the model. The stream lines drawn in Figs. 2 and 3 are symmetric because the lower order of the stream function φ is only considered. Note that at this order the free surface and the interface coincide with their positions at rest.

The downstream waves in the moving frame do not appear in the solutions, obtained above, since these are progressive waves of the same order as that of the plate velocity (second order of ε), and progressive waves of the first order only are calculated here.

The moving plate creates uniform streams propagating faster than the plate itself. Such a result is a reconfirmation for the experimental measurements of Ertekin [2].

Although heavier mathematical calculations are required for approximations of orders higher than the second, the improvement in accuracy is not guaranteed. This is because of the asymptotic nature of the series representations used above for the unknown functions of the problem.

The conclusion is expected to be different for plates moving more rapidly, with velocities varying with time which we hope to be the subject of a forthcoming publication.

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